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On properties of the minus partial order in regular modules

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Abstract. We investigate properties of the minus partial order in regular modules, present new characterizations of this order, and generalize some known results. We introduce a new relation in the general module theoretic setting that is analogous to the space pre-order on complex matrices and study how it is related to the minus partial order.

1. Introduction

Throughout this paper, R denotes an associative ring with identity 1_R, and modules are unitary right R-modules. For a right R-module $M_R = M$, $S = \mathrm{End}_R(M)$ denotes the ring of all right R-module endomorphisms of M. It is well-known that M is a left S- and right R-bimodule. For an (S,R)-bimodule M, let $l_S(.)$ and $r_R(.)$ stand for the left annihilator of a subset of M in S and the right annihilator of a subset of M in R, respectively. If the subset is a singleton, say $\{m\}$, then we simply write $l_S(m)$ and $r_R(m)$, respectively. For a subset $\mathcal{A}$ of a ring R, $l_R(\mathcal{A})$ and $r_R(\mathcal{A})$ denote the left annihilator and the right annihilator of $\mathcal{A}$ in R, respectively. If the subset $\mathcal{A}$ is a singleton, say $\mathcal{A} = \{a\}$, then again we simply write $l_R(a)$ and $r_R(a)$, respectively.

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Let $\mathcal{S}$ be a semigroup and $a \in \mathcal{S}$. Any solution $x = a^-$ to the equation $axa = a$ is called *an inner generalized inverse* of a. If such a^- exists, then a is called *regular,* and if every element in a semigroup $\mathcal{S}$ is regular, then $\mathcal{S}$ is called *a regular semigroup.* Hartwig [3] introduced *the minus partial order* $\leq^-$ on regular semigroups using generalized inverses. For a regular semigroup $\mathcal{S}$ and $a, b \in \mathcal{S}$, we write

$$a \leq^- b \qquad \text{if} \quad a^-a = a^-b \quad \text{and} \quad aa^- = ba^- \tag{1}$$

for some inner generalized inverse a^- of a.

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. For an operator $A \in B(\mathcal{H})$, the symbols Ker A and Im A stand for the kernel and the image of A, respectively. It is known that $A \in B(\mathcal{H})$ is regular if and only if Im $A = \overline{\text{Im } A}$, i.e., the image of A is closed (see, for example, [7]). Šemrl studied in [9] the minus partial order on $B(\mathcal{H})$. He did not want to restrict himself only to operators in $B(\mathcal{H})$ with closed images, so he defined a new order $\leq_S$ on $B(\mathcal{H})$ in the following way: For $A, B \in B(\mathcal{H})$, we write $A \leq_S B$ if there exist idempotent operators $P, Q \in B(\mathcal{H})$ such that Im $P = \overline{\text{Im } A}$, Ker $A =$ Ker Q, $PA = PB$, and $AQ = BQ$. Šemrl called this order the minus partial order on $B(\mathcal{H})$ and proved that this is indeed a partial order on $B(\mathcal{H})$ for a general Hilbert space $\mathcal{H}$. He also showed that the partial order $\leq_S$ is the same as Hartwig's minus partial order $\leq^-$ when $\mathcal{H}$ is finite dimensional.

In [5], *Baer rings* were introduced as rings in which the right (equivalently, left) annihilator of every nonempty subset is generated by an idempotent. To study the torsion theory and motivated by Kaplansky's work on Baer rings, Hattori introduced in [4] principally projective rings. A ring is called *left* (resp. *right*) *principally projective* if every principal left (resp. right) ideal is projective, or equivalently, the left (resp. right) annihilator of any element of the ring is generated by an idempotent as a left (resp. right) ideal. Left (resp. right) principally projective rings are in the literature now usually termed as *left* (resp. *right*) *Rickart rings.* Clearly, every Baer ring is left and right Rickart, i.e., *a Rickart ring.* Note that every Rickart ring has the identity (see, e.g., [1, page 18]). An example of a Baer ring is the algebra $B(\mathcal{H})$.

Following Šemrl's approach, the authors further generalized in [2] the minus partial to Rickart rings. A new relation was introduced in [2] on a ring with identity: Let R be a ring with the multiplicative identity 1_R and $a, b \in R$. Then we write $a \leq^- b$ if there exist idempotent elements $p, q \in R$ such that

$$l_R(a) = R(1_R - p), \quad r_R(a) = (1_R - q)R, \quad pa = pb \quad \text{and} \quad aq = bq. \tag{2}$$

It was proved in [2] that this relation $\leq^{-}$ is indeed a partial order when R is a Rickart ring and that definitions (1) and (2) are equivalent when R is a ring in which every element is regular, i.e., R is a *von Neumann regular ring.* In [10], the present authors introduced a relation on the power set $\mathcal{P}(R)$ of a ring R and showed that this relation, which is called "the minus order on $\mathcal{P}(R)$", is a partial order when R is a Baer ring.

Recently, in [11], the concept of the minus relation has been extended to the module theoretic setting using the endomorphism rings of modules. It was proved that this new relation, which we will present in Section 2, is a partial order when the module is regular.

Motivated by the before-mentioned works on the minus partial order on different structures and, particularly, by the generalization of this notion to the general module theoretic setting, we study in Section 3 properties of the minus partial order on regular modules and obtain some new characterizations of this order. We also introduce a new relation in the general module theoretic setting that is analogous to the space pre-order on complex matrices and study how it is related to the minus partial order.

2. Preliminaries

Let M be a right R-module with $S = \mathrm{End}_R(M)$. For the sake of brevity, in the sequel, S will stand for the endomorphism ring of the module M considered. We will denote the identity map in S by 1_S. The element $m \in M$ is called *a (Zelmanowitz) regular element* if

$$m = m\varphi(m) \equiv m\varphi m$$

for some $\varphi \in M^*$, where $M^* = \mathrm{Hom}_R(M, R)$ denotes the dual of M. Such an element $\varphi \in M^*$ is called *a (Zelmanowitz) regular support* of m and will be denoted by $m^{(1)}$. Also, $\{m^{(1)}\}$ denotes the set of all regular supports of $m \in M$. A module M is called *regular* (in the sense of Zelmanowitz) if every element of M is regular.

For a ring R, let $a \in R$ be a regular element (in the sense of von Neumann). Then there exists $a^{-} \in R$ such that $a = aa^{-}a$. It is well-known that $a \in R$ is regular (in the sense of von Neumann) if and only if a is regular in R_R (or, similarly, in the left R-module ${}_RR$) (in the sense of Zelmanowitz).

Remark 2.1 ([8, Lemma B.47]). Let M be a module and $m \in M$ be regular, say $m = m\varphi m$, where $\varphi \in M^*$. Then $e = \varphi m \in R$ is an idempotent, $mR \cong eR$ is projective, and $M = mR \oplus N$ where $N = \{n \in M : m\varphi n = 0\}$.

Remark 2.2. Let M be a module. It is known that $\mathrm{Hom}_R(R, M) \cong M$. Let $m \in M$ be regular, say $m = m\varphi m$ where $\varphi \in M^*$. Then for the map $m\varphi\colon M \to M$, defined by

$$(m\varphi)(x) = m\varphi(x) \equiv m\varphi x, \quad x \in M,$$

we may conclude that $m\varphi \in S$ and that $m\varphi$ is an idempotent in S.

In [11], the notion of the minus relation was extended to the module theoretic setting. It was proved that the following relation is a partial order when the module M is regular.

Definition 2.3 ([11]). Let M be a module and $m_1, m_2 \in M$. We write $m_1 \leq^- m_2$ if there exists $\varphi \in M^*$ such that $m_1 = m_1\varphi m_1$, $m_1\varphi = m_2\varphi$ and $\varphi m_1 = \varphi m_2$. We call the relation $\leq^-$ the minus order on M.

The following results that were proved in [11] will be used in the continuation.

Proposition 2.4 ([11, Propositions 2.4 and 2.17]). *Let M be a module and $m_1, m_2 \in M$. If $m_1 \leq^- m_2$, then $m_1R \subseteq m_2R$, $l_S(m_2) \subseteq l_S(m_1)$, and $r_R(m_2) \subseteq r_R(m_1)$.*

Proposition 2.5 ([11, see Definition 2.19 and Theorem 2.20]). *Let M be a module, and let $m_1, m_2 \in M$ be regular. If $m_1 \leq^- m_2$, then $m_1R \cap (m_2 - m_1)R = \{0\}$.*

The next result gives a new characterization of the minus (partial) order in (regular) modules.

Proposition 2.6 ([11, Theorem 2.5]). *Let M be a module and $m_1, m_2 \in M$ with m_1 regular. Then $m_1 \leq^- m_2$ if and only if there exist $f^2 = f \in S$, $a^2 = a \in R$ such that $l_S(m_1) = l_S(f)$, $r_R(m_1) = r_R(a)$, $fm_1 = fm_2$, and $m_1a = m_2a$.*

3. The minus partial order

Let M be a module and $m \in M$. We begin with an auxiliary result related to the regular support of m, and then give some characterizations of minus partial order in terms of regular supports.

Lemma 3.1. *Let M be a module. Then we have the following:*

(1) *Let $m \in M$ be a regular element. Then*

$$\{m^{(1)}\} = \{\varphi + (1_R - \varphi m)\alpha + \beta(1_S - m\varphi) : \varphi \in \{m^{(1)}\}, \alpha, \beta \in M^*\}.$$

(2) *Let m_1, m_2 be regular elements of M. Then $m_1 \leq^- m_2$ if and only if there exists $\varphi \in \{m_2^{(1)}\}$ such that $m_1 = m_1\varphi m_2 = m_2\varphi m_1 = m_1\varphi m_1$.*

Proof. (1) Let $K := \{\varphi + (1_R - \varphi m)\alpha + \beta(1_S - m\varphi) : \varphi \in \{m^{(1)}\}, \alpha, \beta, \in M^*\}$ and $x = \varphi + (1_R - \varphi m)\alpha + \beta(1_S - m\varphi) \in K$ for some $\varphi \in \{m^{(1)}\}$ and $\alpha, \beta, \in M^*$. Then

$$mxm = m\varphi m + (m - m\varphi m)\alpha m + m\beta(m - m\varphi m) = m.$$

Hence $x \in \{m^{(1)}\}$, and so $K \subseteq \{m^{(1)}\}$. For the reverse inclusion, let $z \in \{m^{(1)}\}$. Since $z = z + (1_R - zm)0 + 0(1_S - mz) \in K$, we conclude that $\{m^{(1)}\} \subseteq K$.

(2) Assume that $m_1 \leq^- m_2$. By Proposition 2.4, we have $m_1R \subseteq m_2R$. Then there exists $r \in R$ such that $m_1 = m_2r$. Also, since m_2 is regular, there exists $\varphi \in M^*$ such that $m_2 = m_2\varphi m_2$. Hence

$$m_1 = m_2r = m_2\varphi m_2 r = m_2\varphi m_1,$$

and thus $m_1 = m_2\varphi m_1$. Since

$$m_1\varphi m_1 = m_2\varphi m_1 - (m_2 - m_1)\varphi m_1 = m_1 - (m_2 - m_1)\varphi m_1,$$

we have $m_1\varphi m_1 - m_1 = -(m_2 - m_1)\varphi m_1 \in m_1R \cap (m_2 - m_1)R$. By Proposition 2.5, $m_1R \cap (m_2 - m_1)R = \{0\}$, and so $m_1 = m_1\varphi m_1$. Consider

$$m_2 - m_1 = m_2\varphi m_2 - m_2\varphi m_1 = m_2\varphi(m_2 - m_1).$$

Then we have

$$\begin{aligned}(m_2 - m_1)\varphi(m_2 - m_1) &= m_2\varphi(m_2 - m_1) - m_1\varphi(m_2 - m_1)\\ &= (m_2 - m_1) - m_1\varphi(m_2 - m_1),\end{aligned}$$

and therefore

$$(m_2 - m_1) - (m_2 - m_1)\varphi(m_2 - m_1) = m_1\varphi(m_2 - m_1) \in m_1R \cap (m_2 - m_1)R.$$

Again, since $m_1R \cap (m_2 - m_1)R = \{0\}$, we may conclude that $m_1 = m_1\varphi m_1 = m_1\varphi m_2$. Therefore, we have $m_1 = m_1\varphi m_2 = m_2\varphi m_1 = m_1\varphi m_1$.

Conversely, suppose there exists $\varphi \in \{m_2^{(1)}\}$ such that $m_1 = m_1\varphi m_2 = m_2\varphi m_1 = m_1\varphi m_1$. Let $\alpha := \varphi m_1 \varphi \in M^*$. On the one hand, we have

$$m_1\alpha m_1 = m_1\varphi m_1\varphi m_1 = m_1\varphi m_1 = m_1,$$

and on the other hand, we obtain

$$m_1\alpha = m_1\varphi m_1\varphi = m_1\varphi = m_2\varphi m_1\varphi = m_2\alpha$$

and

$$\alpha m_1 = \varphi m_1\varphi m_1 = \varphi m_1 = \varphi m_1\varphi m_2 = \alpha m_2.$$

Therefore, by Definition 2.3, we may conclude that $m_1 \leq^- m_2$. □

Theorem 3.2. *Let M be a regular module and $m_1, m_2 \in M$. Then the following are equivalent:*

(1) $m_1 \leq^- m_2$;

(2) $\{m_1^{(1)}\} \cap \{m_2^{(1)}\} \neq \emptyset$, $m_1R \subseteq m_2R$, *and* $Sm_1 \subseteq Sm_2$;

(3) $\{m_2^{(1)}\} \subseteq \{m_1^{(1)}\}$.

Proof. (1) ⇒ (2) It follows directly from Lemma 3.1(2).

(2) ⇒ (3) Assume that $\{m_1^{(1)}\}\cap\{m_2^{(1)}\} \neq \emptyset$, $m_1R \subseteq m_2R$, and $Sm_1 \subseteq Sm_2$. Then $m_1 = fm_2 = m_2r$ for some $f \in S$ and $r \in R$. Let $x \in \{m_2^{(1)}\}$, i.e., $m_2 = m_2xm_2$. Since $\{m_1^{(1)}\} \cap \{m_2^{(1)}\} \neq \emptyset$, there exists $\alpha \in M^*$ such that $m_1 = m_1\alpha m_1$ and $m_2 = m_2\alpha m_2$. On the one hand,

$$m_1xm_1 = fm_2xm_2r = fm_2r = fm_1,$$

and on the other hand, $m_1 = m_1\alpha m_1 = fm_2\alpha m_2r = fm_2r = fm_1$, and so $m_1 = m_1xm_1$. Hence $x \in \{m_1^{(1)}\}$, and thus $\{m_2^{(1)}\} \subseteq \{m_1^{(1)}\}$.

(3) ⇒ (1) Assume that $\{m_2^{(1)}\} \subseteq \{m_1^{(1)}\}$. Note that m_2 is regular, and let $\varphi \in \{m_2^{(1)}\}$. By assumption, $m_1\varphi m_1 = m_1$. We will prove that $m_1 = m_1\varphi m_2$ and $m_1 = m_2\varphi m_1$. Let $K_\varphi := \{\varphi + (1_R - \varphi m_2)\alpha + \beta(1_S - m_2\varphi) : \alpha, \beta \in M^*\}$ where φ is as above. For any $\alpha \in K_\varphi$, Lemma 3.1(1) implies that $\alpha \in \{m_2^{(1)}\}$. Fix for a while an arbitrary $\alpha \in K_\varphi$. Then there exist $\gamma, \beta \in M^*$ such that

$$\alpha = \varphi + (1_R - \varphi m_2)\gamma + \beta(1_S - m_2\varphi). \tag{3}$$

Observe that $\varphi, \alpha \in \{m_2^{(1)}\} \subseteq \{m_1^{(1)}\}$ implies $m_1\varphi m_1 = m_1\alpha m_1$. Multiplying (3) by m_1 from the left and the right we thus get

$$(m_1 - m_1\varphi m_2)\gamma m_1 + m_1\beta(1_S - m_2\varphi)m_1 = 0. \tag{4}$$

Since $\alpha \in K_\varphi$ was arbitrary, (4) holds for any γ and β in M^*. By taking $\gamma = 0$, we thus have $m_1M^*(1_S - m_2\varphi)m_1 = \{0\}$. Then $(1_S - m_2\varphi)m_1M^*(1_S - m_2\varphi)m_1 = \{0\}$. Let $x = (1_S - m_2\varphi)m_1 \in M$. Since M is a regular module, there exists $g \in M^*$ with $xgx = x$. So, $xgx = x$ and $xM^*x = \{0\}$, which implies $x = 0$. It follows that $m_1 = m_2\varphi m_1$. Similarly, by taking $\beta = 0$, we get $(m_1 - m_1\varphi m_2)M^*m_1 = \{0\}$, from which we may conclude that $m_1 = m_1\varphi m_2$. It follows that $m_1 \leq^- m_2$ by Lemma 3.1(2). □

As a consequence of Proposition 2.4, Lemma 3.1(2) and (1) ⇔ (2) in Theorem 3.2, we obtain the following characterization for the minus partial order.

Corollary 3.3. *Let M be a regular module and $m_1, m_2 \in M$. Then the following are equivalent:*

(1) $m_1 \leq^- m_2$;

(2) $l_S(m_2) \subseteq l_S(m_1)$, $r_R(m_2) \subseteq r_R(m_1)$, *and there exists $\alpha \in M^*$ such that* $m_1 = m_1\alpha m_1$, $m_2 = m_2\alpha m_2$.

Proof. (1) ⇒ (2) It follows immediately by Proposition 2.4 and (1) ⇔ (2) in Theorem 3.2.

(2) ⇒ (1) Assume that $l_S(m_2) \subseteq l_S(m_1)$, $r_R(m_2) \subseteq r_R(m_1)$, and $m_1 = m_1\alpha m_1$, $m_2 = m_2\alpha m_2$ for some $\alpha \in M^*$. Let $f = m_2\alpha \in S$ and $a = \alpha m_2 \in R$. Then $1_S - f \in l_S(m_2)$ and $1_R - a \in r_R(m_2)$, and thus $1_S - f \in l_S(m_1)$ and $1_R - a \in r_R(m_1)$ by assumption. Hence $m_1 = fm_1 = m_2\alpha m_1$ and $m_1 = m_1a = m_1\alpha m_2$, and therefore $m_1 \leq^- m_2$ by Lemma 3.1(2). □

For a module M and $m \in M$, let

$$D_1(m) := \{\varphi - \alpha : \varphi, \alpha \in \{m^{(1)}\}\}.$$

Lemma 3.4. *Let m be a regular element in a module M. Then*

$$D_1(m) = \{\beta \in M^* : m\beta m = 0\}.$$

Proof. Let $m \in M$ be regular, and let $A := \{\beta \in M^* : m\beta m = 0\}$. If $\varphi - \alpha \in D_1(m)$, then

$$m(\varphi - \alpha)m = m\varphi m - m\alpha m = m - m = 0.$$

Hence $\varphi - \alpha \in A$. Conversely, if $\beta \in A$, then $m\beta m = 0$. Since m is regular, there exists $\sigma \in M^*$ such that $m = m\sigma m$. Thus $m(\sigma - \beta)m = m\sigma m - m\beta m = m$, and hence $\sigma - \beta \in \{m^{(1)}\}$. Since $\sigma \in \{m^{(1)}\}$, $\sigma - (\sigma - \beta) = \beta \in D_1(m)$. To conclude, $D_1(m) = \{\beta \in M^* : m\beta m = 0\}$. □

Theorem 3.5. *Let M be a regular module and $m_1, m_2 \in M$. Then the following are equivalent:*

(1) $m_1 \leq^- m_2$;

(2) $\{m_2^{(1)}\} \cap \{m_1^{(1)}\} \neq \emptyset$ *and* $D_1(m_2) \subseteq D_1(m_1)$.

Proof. (1) $\Rightarrow$ (2) By Theorem 3.2, we have $\{m_2^{(1)}\} \subseteq \{m_1^{(1)}\}$. Since $m_2 \in M$ is regular, there exists $\varphi \in M^*$ such that $m_2 = m_2\varphi m_2$. It follows, $\varphi \in \{m_2^{(1)}\} \cap \{m_1^{(1)}\}$ and so $\{m_2^{(1)}\} \cap \{m_1^{(1)}\} \neq \emptyset$. Let now $\beta \in D_1(m_2)$. By the definition of $D_1(m_2)$, there exists $\varphi, \alpha \in \{m_2^{(1)}\} \subseteq \{m_1^{(1)}\}$ such that $\beta = \varphi - \alpha$, and so $\beta \in D_1(m_1)$. Hence $D_1(m_2) \subseteq D_1(m_1)$.
(2) $\Rightarrow$ (1) Since $\{m_2^{(1)}\} \cap \{m_1^{(1)}\} \neq \emptyset$, there exists $\varphi \in M^*$ such that $\varphi \in \{m_2^{(1)}\} \cap \{m_1^{(1)}\}$. We will show that $m_1 = m_1\varphi m_2 = m_2\varphi m_1 = m_1\varphi m_1$. From $\varphi \in \{m_2^{(1)}\} \cap \{m_1^{(1)}\}$ we get $m_1 = m_1\varphi m_1$ and $m_2 = m_2\varphi m_2$. Hence, on the one hand,

$$m_2(1_R - \varphi m_2)M^* m_2 = \{0\},$$

and so by Lemma 3.4, $(1_R - \varphi m_2)M^* \subseteq D_1(m_2) \subseteq D_1(m_1)$. Again, by Lemma 3.4, $m_1(1_R - \varphi m_2)M^* m_1 = \{0\}$, and thus $m_1(1_R - \varphi m_2)M^* m_1(1_R - \varphi m_2) = \{0\}$. Let $x := m_1(1_R - \varphi m_2)$. It follows that $xM^*x = \{0\}$. Regularity of M implies that there exists $\varphi_x \in M^*$ such that $x = x\varphi_x x$. Hence $0 = x\varphi_x x = x$, and so $m_1 = m_1\varphi m_2$. On the other hand,

$$m_2 M^*(1_S - m_2\varphi)m_2 = \{0\},$$

and so $M^*(1_S - m_2\varphi) \subseteq D_1(m_2) \subseteq D_1(m_1)$. By Lemma 3.4, $m_1 M^*(1_S - m_2\varphi)m_1 = \{0\}$. Similarly, regularity of M implies that $m_1 = m_2\varphi m_1$. We may conclude by Lemma 3.1(2) that $m_1 \leq^- m_2$. □

Now we are going to define a new relation on modules which is analogous to the definition of the space pre-order on complex matrices introduced by Mitra in [6].

Definition 3.6. Let M be a module and $m_1, m_2 \in M$. We write $m_1 \leq_S m_2$ if $Sm_1 \subseteq Sm_2$ and $m_1R \subseteq m_2R$. We call the relation $\leq_S$ the space pre-order on M.

With the next result we will show that the minus order implies the space pre-order.

Theorem 3.7. *Let M be a module and $m_1, m_2 \in M$. If $m_1 \leq^- m_2$, then $m_1 \leq_S m_2$.*

Proof. Let $m_1 \leq^- m_2$. By Proposition 2.6, there exist $f^2 = f \in S$ and $a^2 = a \in R$ such that $l_S(m_1) = l_S(f)$, $r_R(m_1) = r_R(a)$, $fm_1 = fm_2$, and $m_1 a = m_2 a$. Since $m_1 = m_1 a = m_2 a \in m_2 R$, we have $m_1 R \subseteq m_2 R$. From $m_1 = fm_1 = fm_2 \in Sm_2$ we obtain $Sm_1 \subseteq Sm_2$. □

The converse statement of Theorem 3.7 is not true in general as the following example shows.

Example 3.8. Let $\mathbb{Z}_6$ denote the ring of integers modulo 6, and consider the ring $\mathbb{Z}_6$ as a module over itself. For any ring R, since $\mathrm{Hom}_R(R,R) \cong R$, we may take $M = R = S = \mathbb{Z}_6$, and also $M^* = \mathbb{Z}_6$. Let $m_1 = \overline{4}$ and $m_2 = \overline{2}$ in M. Since $\overline{4} = \overline{4} \cdot \overline{4} \cdot \overline{4}$ and $\overline{2} = \overline{2} \cdot \overline{2} \cdot \overline{2}$, m_1 and m_2 are regular elements of M. Note that $\mathbb{Z}_6$ is a commutative ring. From $\overline{4}\mathbb{Z}_6 = \overline{2}\mathbb{Z}_6$ we obtain $Sm_1 = Sm_2$ and $m_1 R = m_2 R$. Therefore $m_1 \leq_S m_2$. Note that all of the idempotents of $\mathbb{Z}_6$ are $\overline{0}, \overline{1}, \overline{3}$ and $\overline{4}$. Observe also $\overline{4}$ is the only idempotent of $\mathbb{Z}_6$ such that $l_S(m_1) = l_S(f)$ and $r_R(m_1) = r_R(a)$ where $f^2 = f \in S$ and $a^2 = a \in R$, i.e., $f = a = \overline{4}$. But $\overline{4} \cdot \overline{4} \neq \overline{4} \cdot \overline{2}$, and so $fm_1 \neq fm_2$ and also $m_1 a \neq m_2 a$. Therefore, $m_1 \not\leq^- m_2$.

We will now present new characterizations of the space pre-order on M where M is a regular module.

Theorem 3.9. *Let M be a regular module and $m_1, m_2 \in M$. Then the following are equivalent:*

(1) $m_1 \leq_S m_2$;

(2) $l_S(m_2) \subseteq l_S(m_1)$ *and* $r_R(m_2) \subseteq r_R(m_1)$;

(3) $m_1 = m_1 \varphi m_2 = m_2 \varphi m_1$ *for all* $\varphi \in \{m_2^{(1)}\}$;

(4) $m_1 D_1(m_2) m_1 = \{0\}$.

Proof. (1) ⇒ (2) Clear.

(2) ⇒ (3) Since M is regular, there exists $\varphi \in M^*$ such that $m_2 = m_2 \varphi m_2$. Then $m_2(1_R - \varphi m_2) = 0$ and so $1_R - \varphi m_2 \in r_R(m_2) \subseteq r_R(m_1)$. Hence we have $m_1(1_R - \varphi m_2) = 0$, and thus $m_1 = m_1 \varphi m_2$. Also, $(1_S - m_2 \varphi) m_2 = 0$, and so $1_S - m_2 \varphi \in l_S(m_2) \subseteq l_S(m_1)$. It follows that $(1_S - m_2 \varphi) m_1 = 0$, which yields $m_1 = m_2 \varphi m_1$.

(3) ⇒ (4) Let $x \in D_1(m_2)$. Then there exist $\varphi, \alpha \in \{m_2^{(1)}\}$ such that $x = \varphi - \alpha$. Since $m_1 \varphi m_1 = m_1 \varphi m_2 \alpha m_1 = m_1 \alpha m_1$,

$$m_1 x m_1 = m_1(\varphi - \alpha) m_1 = m_1 \varphi m_1 - m_1 \alpha m_1 = 0.$$

Therefore, $m_1 D_1(m_2) m_1 = \{0\}$.

(4) $\Rightarrow$ (1) Assume that $m_1 D_1(m_2) m_1 = \{0\}$. Since M is regular, there exists $\varphi \in M^*$ such that $m_2 = m_2 \varphi m_2$. Then $m_2(1_R - \varphi m_2) = 0$, and so $m_2(1_R - \varphi m_2)\alpha = 0$ for any $\alpha \in M^*$. Since $m_2(1_R - \varphi m_2)\alpha m_2 = 0$, $(1_R - \varphi m_2)\alpha \in D_1(m_2)$. By assumption, we have $m_1(1_R - \varphi m_2)\alpha m_1 = 0$, and hence

$$[m_1(1_R - \varphi m_2)]\alpha[m_1(1_R - \varphi m_2)] = 0.$$

Regularity of M implies that $m_1(1_R - \varphi m_2) = 0$, and so $m_1 = m_1 \varphi m_2$. Thus $Sm_1 \subseteq Sm_2$. Similarly, we can obtain $m_1 = m_2 \varphi m_1$ and so $m_1 R \subseteq m_2 R$. Hence $m_1 \leq_S m_2$. □

The next result follows directly from Theorem 3.9 ((1) $\Leftrightarrow$ (4)) and Lemma 3.4.

Corollary 3.10. *Let M be a regular module and $m_1, m_2 \in M$. Then $m_1 \leq_S m_2$ if and only if $D_1(m_2) \subseteq D_1(m_1)$.*

As a direct consequence of Lemma 3.1(2) and (1) $\Leftrightarrow$ (3) in Theorem 3.9, we obtain the following characterization for the minus partial order on regular modules. Moreover, with the next result, with which we conclude the paper, we introduce a condition under which the converse statement of Theorem 3.7 holds.

Corollary 3.11. *Let M be a regular module and $m_1, m_2 \in M$. Then the following are equivalent:*

(1) $m_1 \leq^- m_2$;

(2) $m_1 \leq_S m_2$ *and* $\{m_2^{(1)}\} \cap \{m_1^{(1)}\} \neq \emptyset$.

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