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Integral and homothetic indecomposability with applications to irreducibility of polynomials

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Abstract

Being motivated by some methods for construction of homothetically indecomposable polytopes, we obtain new methods for construction of families of integrally indecomposable polytopes. As a result, we find new infinite families of absolutely irreducible multivariate polynomials over any field F. Moreover, we provide different proofs of some of the main results of Gao [2].

Key Words: Absolute irreducibility, polytopes, integral indecomposability.

1. Introduction

Absolutely irreducible polynomials over a field are important and have applications in many areas of algebra and geometry such as algebraic geometry, number theory, coding theory, combinatorics, permutation polynomials. We have some well-known irreducibility criteria such as Eisenstein criterion and Eisenstein-Dumas criterion. Another criterion in the literatures is known as Newton polygon method. Recently, Gao has strengthened Newton polygon method as *Newton polytope method.* As a result of Newton polytope method, existence of an integrally indecomposable polytope in $\mathbb{R}^n$ implies the existence of an infinite family of absolutely irreducible polynomials in n variables over an arbitrary field (see Remark 1.4 below). In [2, 3], infinite classes of integrally indecomposable polytopes have been found by some methods for constructing integrally indecomposable polytopes. A connection of homothetic indecomposability to integral indecomposability was also given in [3].

In this study we obtain further connections and applications of homothetic indecomposability to integral indecomposability. Using modifications of some methods for construction of homothetically indecomposable polytopes, we obtain new methods for construction of new families of integrally indecomposable polytopes, which are not included in [2, 3]. Therefore, we find new infinite families of absolutely irreducible polynomials (see Theorems 2.4, 2.7 and 2.8 below). Moreover, using our results we give different proofs of some of the main results of [2], see Example 2.10 below. Throughout the paper, we provide concrete examples illustrating our results.

In the rest of this section we give some basic facts and definitions for Newton polytope method. We give our results in Section 2, after some background on the homothetic indecomposability.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and S be a subset of $\mathbb{R}^n$. The smallest convex set containing S, denoted by *conv(S)*, is called the *convex hull* of S. When $S = \{a_1, a_2, ..., a_n\}$ is a finite set, we shall write $conv(a_1, ..., a_n)$ instead of $conv(S)$. It can be shown easily that

$$conv(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \{x_1, ..., x_k\} \subseteq S, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The *affine hull* $\operatorname{aff}(S)$ of S is defined as

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \{x_1, ..., x_k\} \subseteq S, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

For any point $x \in S$, x is said to be in the *relative interior* of S, denoted as $x \in \operatorname{relint}(S)$, if x lies in the interior of S relative to $\operatorname{aff}(S)$, i.e. there exists an open ball B in $\operatorname{aff}(S)$ such that $x \in B \subset S$.

The convex hull of finitely many points in $\mathbb{R}^n$ is called a ***polytope***. A point of a polytope is called a *vertex* if it is not on the line segment joining any other two different points of the polytope. It is known that a polytope is always the convex hull of its vertices, for example see [13, Proposition 2.2].

The main operation for convex sets in $\mathbb{R}^n$ is defined as follows.

Definition 1.1 For any two sets A and B in $\mathbb{R}^n$, the sum

$$A + B = \{a + b : a \in A, b \in B\}$$

is called *Minkowski sum*, or *vector addition* of A and B.

A point in $\mathbb{R}^n$ is called *integral* if its coordinates are integers. A polytope in $\mathbb{R}^n$ is called *integral* if all of its vertices are integral. An integral polytope C is called ***integrally decomposable*** if there exist integral polytopes A and B such that $C = A + B$ where both A and B have at least two points. Otherwise, C is called ***integrally indecomposable***.

Let F be any field and consider any polynomial

$$f(x_1, x_2, ..., x_n) = \sum c_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n} \in F[x_1, ..., x_n].$$

We can think an exponent vector $(e_1, e_2, ..., e_n)$ of f as a point in $\mathbb{R}^n$. The ***Newton polytope*** of f, denoted by P_f, is defined as the convex hull in $\mathbb{R}^n$ of all the points $(e_1, ..., e_n)$ with $c_{e_1 e_2 ... e_n} \neq 0$.

Recall that a polynomial over a field F is called ***absolutely irreducible*** if it remains irreducible over every algebraic extension of F.

By using Newton polytopes of multivariate polynomials, we can determine infinite families of absolutely irreducible polynomials over an arbitrary field F by using the following result due to Ostrowski [9].

Lemma 1.2 *Let $f, g, h \in F[x_1, ..., x_n]$ with $f \neq 0$ and $f = gh$. Then $P_f = P_g + P_h$.*

Proof. See, for example, the proof of [2, Lemma 2.1]. □

As a direct result of Lemma 1.2, we have the following corollary which is an *irreducibility criterion* for multivariate polynomials over arbitrary fields.

Corollary 1.3 *Let F be any field and f a nonzero polynomial in $F[x_1, ..., x_n]$ not divisible by any x_i. If the Newton polytope P_f of f is integrally indecomposable then f is absolutely irreducible over F.*

Proof. See [2, page 507]. □

Remark 1.4 *For a polynomial $f \in F[x_1, x_2, ..., x_n]$, if P_f is integrally indecomposable, we say that f is absolutely irreducible over F by the polytope method.*

Notation: For any element $v = (a_1, ..., a_n)$ of $\mathbb{Z}^n$ we shall write $gcd(v)$ to mean $gcd(a_1, ..., a_n)$, i.e. the greatest common divisor of all the components of v. Similarly, for several vectors $v_1, ..., v_k$ in $\mathbb{Z}^n$, by writing $gcd(v_1, ..., v_k)$ we mean the greatest common divisor of all the components of the vectors $v_1, ..., v_k$. For any points $v_1, v_2 \in \mathbb{Z}^n$, $[v_1, v_2]$ refers to line segment from v_1 to v_2.

2. Integrally indecomposable polytopes in $\mathbb{R}^n$

Beside the integral indecomposability, there is another concept, *homothetic indecomposability* for polytopes; see the book [4, Chapter 15]. Let P and Q be polytopes in $\mathbb{R}^n$, not necessarily integral. Q is said to be homothetic to P if there exists a real number $r \geq 0$ and a vector $v \in \mathbb{R}^n$ such that

$$Q = rP + v = \{ra + v : a \in P\}.$$

A polytope Q is said to be homothetically indecomposable if $Q = A + B$ for some polytopes A and B then either A or B is homothetic to Q, e.g. if A is homothetic to Q then

$$Q = A + B = (rQ + v) + (1 - r)Q + (-v)$$

for some $0 \leq r \leq 1$ and $v \in \mathbb{R}^n$. Otherwise, Q is called homothetically decomposable. Homothetically indecomposable polytopes have been widely studied in the literature, for example in [5, 7, 8, 10, 11, 12].

There is no direct comparison between integral and homothetic indecomposability of polytopes. A polytope may satisfy only one of them or both or none.

For example, the only homothetically indecomposable polytopes in the plane are line segments and triangles. Any summand of a line segment must be parallel to it and have smaller length than itself. Also, the edges of a summand of triangle T must be parallel to the edges of T and have smaller length than them.

As we know, only some triangles or line segments, and many polygons having more than three edges are integrally indecomposable. An integral square is both integrally and homothetically decomposable. There is a result in [3] giving a relation between these two different concepts of decomposability of polytopes.

Proposition 2.1 *Let P be an integral polytope in $\mathbb{R}^n$ with vertices $v_1, ..., v_m$. If P is homothetically indecomposable and*

$$gcd(v_1 - v_2, ..., v_1 - v_m) = 1$$

then P is integrally indecomposable.

Proof. See the proof of [3, Proposition 12]. □

From Proposition 2.1 we get the following simple and useful lemma.

Lemma 2.2 *Let Q be a homothetically indecomposable integral polytope with vertices $v_1, \ldots, v_m$. Then, Q is integrally indecomposable if and only if*

$$gcd(v_1 - v_2, ..., v_1 - v_m) = 1.$$

Proof. Let $gcd(v_1 - v_2, ..., v_1 - v_m) = d > 1$. Then, the polytope $P = conv(0, v_2 - v_1, ..., v_m - v_1)$ is integral. Therefore, $Q = v_1 + d \cdot (\frac{1}{d}P)$.

Converse follows from Proposition 2.1. □

By Lemma 2.2, we can get infinitely many integrally indecomposable polytopes using the homothetically indecomposable polytopes constructed in [5, 7, 8, 10, 11, 12].

In this section, we need some new terminologies. For details, see [1].

Definition 2.3 *For $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^n$ the set*

$$H = \{x \in \mathbb{R}^n : \beta \cdot x = \alpha\}$$

is called a hyperplane, where

$$\beta \cdot x = \beta_1 v_1 + ... + \beta_n v_n$$

is the dot product of the vectors $\beta = (\beta_1, ..., \beta_n)$, $v = (v_1, ..., v_n)$. In a natural manner, the closed halfspaces formed by H are defined as

$$H^- = \{x \in \mathbb{R}^n : \beta \cdot x \leq \alpha\}, \quad H^+ = \{x \in \mathbb{R}^n : \beta \cdot x \geq \alpha\}.$$

A hyperplane H_K is called a supporting hyperplane of a closed convex set $K \subset \mathbb{R}^n$ if $K \subset H_K^+$ or $K \subset H_K^-$ and $K \cap H_K \neq \emptyset$, i.e. H_K contains a boundary point of K. A supporting hyperplane H_K of K is called nontrivial if K is not contained in H_K. The halfspace H_K^- (or H_K^+) is called a supporting halfspace of K, possibly we may have $K \subset H_K$.

Let $C \subset \mathbb{R}^n$ be a compact convex set. Then for any nonzero vector $v \in \mathbb{R}^n$, the real number $s = sup_{x \in C}(x \cdot v)$ is defined as $max\{x \cdot v : x \in C\}$, where

$$x \cdot v = x_1 v_1 + ... + x_n v_n$$

is the dot product of the vectors $x = (x_1, ..., x_n)$ *and* $v = (v_1, ..., v_n)$.

Let $K \subset \mathbb{R}^n$ *be a nonempty convex compact set. The map*

$$h_K : \mathbb{R}^n \to \mathbb{R}, \qquad u \to sup_{x \in K}(x \cdot u)$$

is called the support function of K.

Let $K \subset \mathbb{R}^n$ *be a nonempty convex compact set. For every fixed nonzero vector* $u \in \mathbb{R}^n$, *the hyperplane having normal vector* u *is defined as*

$$H_K(u) = \{x \in \mathbb{R}^n : x \cdot u = h_K(u)\}.$$

Note that $H_K(u)$ *is a supporting hyperplane of* K.

It is known that every supporting hyperplane of K *has a representation of this form. See [1, Page 19].*

Let P *be a polytope. The intersection of* P *with a supporting hyperplane* H_P *is called a face of* P. *A vertex of* P *is a face of dimension zero. An edge of* P *is a face of dimension* 1, *which is a line segment. A face* F *of* P *is called a facet if dim (F)= dim (P)* -1. *If* u *is any nonzero vector in* $\mathbb{R}^n$, $F_P(u) = H_P(u) \cap P$ *shows the face of* P *in the direction of* u, *that is the intersection of* P *with its supporting hyperplane* $H_P(u)$ *having outer normal vector* u. *It is known that* $F_P(u) = F_Q(u) + F_R(u)$ *if* $P = Q + R$ *for some polytopes* Q *and* R.

Let P be a polytope. A sequence $F_0, F_1, ..., F_m$ of faces of P is called a *strong chain* if $dim(F_i \cap F_{i+1}) \geq 1$ for $i = 0, ..., m-1$. Such a chain is said to join two vertices u and v of P if, say $u \in F_0$, and $v \in F_m$. See [7].

We observe that in the theorems about homothetic indecomposability given in [10, 7], we can consider strong chain of integrally indecomposable faces instead of strong chain of homothetically indecomposable faces. As a result, we get new theorems giving infinitely many new integrally indecomposable polytopes in $\mathbb{R}^n$.

Theorem 2.4 *Let* P *be an integral polytope in* $\mathbb{R}^n$ *such that any two of its vertices can be joined by a strong chain of integrally indecomposable faces. Then* P *is integrally indecomposable.*

Proof. Let us assume that $P = Q + R$ for some integral polytopes such that $P = conv(p_1, p_2, ..., p_m)$ and $Q = conv(q_1, q_2, ..., q_m)$, where in order to have the same number of vertices we allow the repetition of vertices of Q. We shall show that Q is a translation of P, i.e. $Q = P + v$ for some vector $v \in \mathbb{R}^n$.

Let p_i be any vertex of P and $F = conv(p_i, p_{i+1}, ..., p_k) = P \cap H_P(u)$ be an integrally indecomposable face of P containing p_i, where $H_P(u)$ is a supporting hyperplane of P having normal vector $u \in \mathbb{R}^n$. Then, the corresponding face $G = conv(q_i, q_{i+1}, ..., q_k) = F \cap H_P(u)$ of Q, i.e. $F = G + H$ for some face H of R, must be of the form $G = F + v$ for some vector $v \in \mathbb{R}^n$. Thus, any edge $[q_j, q_{j+1}]$ of Q must be of the form $[q_j, q_{j+1}] = [p_j, p_{j+1}] + v$. Hence, $q_j = p_j + v$ and $q_{j+1} = p_{j+1} + v$. Since any two vertices e, e' of P can be joined by a strong chain of integrally indecomposable faces $F_1, ..., F_s$, where $e \in F_1, ..., e' \in F_s$ and $dim\ (F_i \cap F_{i+1}) \geq 1$ we conclude that $q_i = p_i + v$ for all $i, = 1, ..., m$. So, $Q = P + v$. Consequently, P is integrally indecomposable. □

Example 2.5 *As a result of Theorem 2.4, we give six new integrally indecomposable polytopes in* $\mathbb{R}^n$. *We can have infinitely many examples of this kind by taking extra suitable hyperplanes in the following items. We* *consider our polytopes as seen in the respective figures. That is, in Figures 1,2, 4, 5 and 6, we assume that projection of* C_1 *on* C_2 *lies in* $relint(C_2)$.

(1) *Consider Figure 1. Let* $n \geq 3$ *be an integer. Let* $C_1 = conv(v_1, v_2, v_3, ..., v_n)$, $C_2 = conv(u_1, u_2, u_3, ..., u_{2n})$ *and* $C_3 = conv(w_1, w_2, w_3, ..., w_n)$ *be integral polytopes lying on different nonparallel hyperplanes as shown in Figure 1. Consider the integral polytope*

$$C = conv(C_1, C_2, C_3).$$

Assume that lateral white faces of C *are integrally indecomposable quadrangles* $conv(v_1, v_2, u_2, u_3)$, $conv(v_2, v_3, u_4, u_5), ...,$ $conv(v_n, v_1, u_{2n}, u_1)$. *Then,* C *is integrally indecomposable if* C_1 *or* C_2 *is integrally indecomposable.*

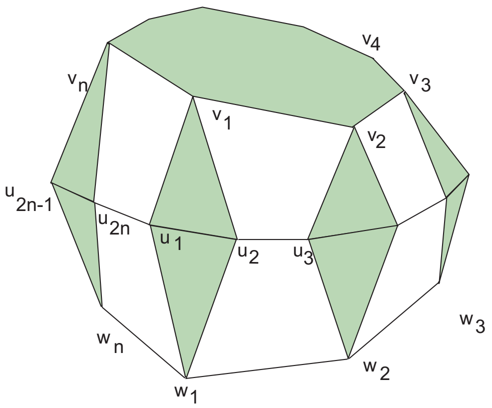

Figure 1.

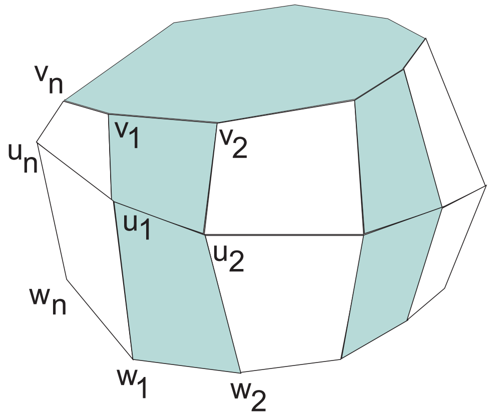

Figure 2.

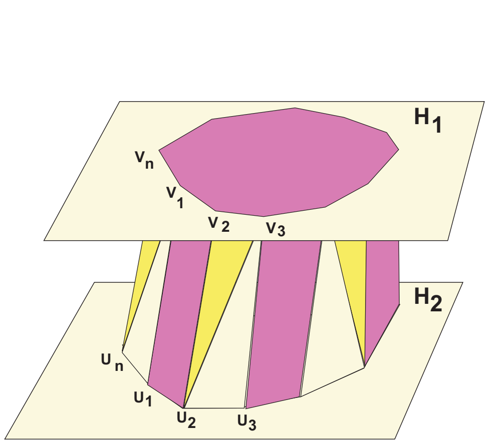

Figure 3.

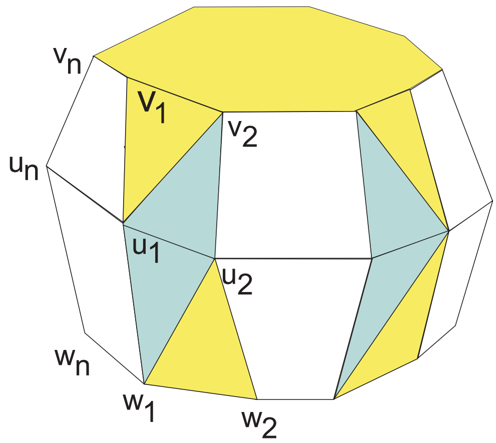

Figure 4.

(2) *Consider Figure 2. Let* $n \geq 4$ *be an even integer. We take the integral polytopes* $C_1 = conv(v_1, v_2, v_3, ..., v_n)$, $C_2 = conv(u_1, u_2, u_3, ..., u_n)$ *and* $C_3 = conv(w_1, w_2, w_3, ..., w_n)$ *lying on different nonparallel hyperplanes*

as shown in Figure 2. Consider the integral polytope

$$C = conv(C_1, C_2, C_3).$$

Suppose that the lateral faces $conv(v_1, v_2, u_1, u_2)$, $conv(v_3, v_4, u_3, u_4), \ldots,$ $conv(v_{n-1}, v_n, u_{n-1}, u_n)$ *are integrally indecomposable quadrangles. If* C_1 *or* C_2 *is integrally indecomposable then,* C *is integrally indecomposable.*

(3) *Consider Figure 3. Let* $n \geq 4$ *be an integer and* H_1, H_2 *be parallel nonintersecting hyperplanes in* $\mathbb{R}^n$. *Let* $C_1 = conv(v_1, v_2, v_3, \ldots, v_n) \subset H_1$ *and* $C_2 = conv(u_1, u_2, u_3, \ldots, u_n) \subset H_2$ *be integral polytopes such that*

(i) $[v_1, v_2]$ *is not parallel to* $[u_1, u_2]$, $[v_3, v_4]$ *is not parallel to* $[u_3, u_4], \ldots,$ *(*$[v_n, v_1]$ *is not parallel to* $[u_n, u_1]$ *if* n *is a positive odd integer),*

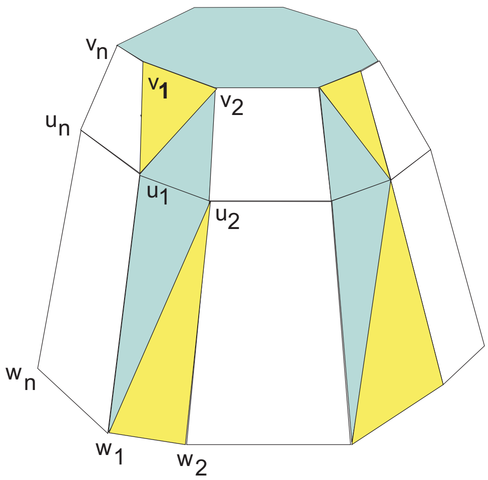

Figure 5.

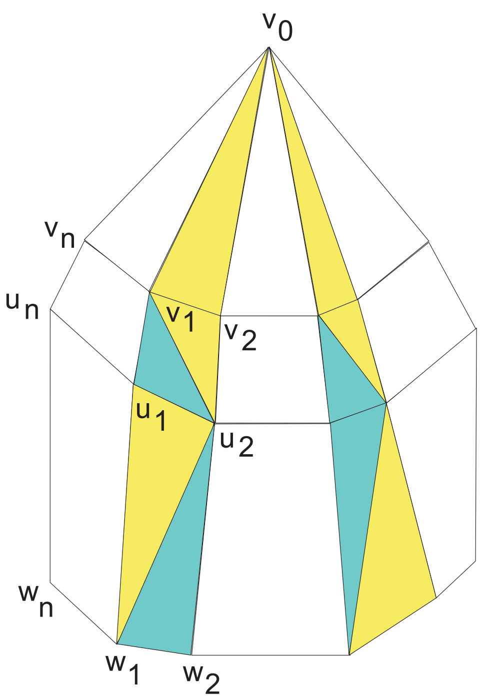

Figure 6.

(ii) $[v_2, v_3]$ *is not parallel to* $[u_2, u_3]$, $[v_4, v_5]$ *is not parallel to* $[u_4, u_5], \ldots,$ *(*$[v_n, v_1]$ *is not parallel to* $[u_n, u_1]$ *if* n *is a positive even integer).*

Figure 3 corresponds to the case n *is an even positive integer. Assume that the triangular lateral faces of the polytope* $C = conv(C_1, C_2)$ *are integrally indecomposable. If* C_1 *or* C_2 *is integrally indecomposable then so is* C*. Alternatively, if two adjacent triangular faces, the face* C_1 *and the face* C_2 *are integrally indecomposable then* C *is integrally indecomposable.*
Note that for non-parallel hyperplanes H_1 *and* H_2 *in* $\mathbb{R}^n$, *a similar result holds.*

(4) *Consider Figure 4. Let* $n \geq 4$ *be an integer. Let* $C_1 = conv(v_1, v_2, v_3, ..., v_n)$, $C_2 = conv(u_1, u_2, u_3, ..., u_n)$, *and* $C_3 = conv(w_1, w_2, w_3, ..., w_n)$ *be integral polytopes lying on different parallel hyperplanes as shown in Figure 4. Consider the integral polytope* $C = conv(C_1, C_2, C_3)$. *Assume that*

(i) $[v_1, v_2]$ *is not parallel to* $[u_1, u_2]$, $[v_3, v_4]$ *is not parallel to* $[u_3, u_4], ...,$ *(*$[v_{n-1}, v_n]$ *is not parallel to* $[u_{n-1}, u_n]$ *if* n *is a positive even integer),*

(ii) $[u_1, u_2]$ *is not parallel to* $[w_1, w_2]$, $[u_3, u_4]$ *is not parallel to* $[w_3, w_4], ...,$ *(*$[u_n, u_1]$ *is not parallel to* $[w_n, w_1]$ *if* n *is a positive odd integer).*

Also suppose that the lateral triangular faces of C *are integrally indecomposable. Then,* C *is integrally indecomposable if* C_1 *or* C_2 *is integrally indecomposable.*

(5) *Consider the polytope* P *in Figure 5.* P *is in the type of the polytope of case (4). Therefore, the same result holds in this case also. Furthermore, in this case* P *lies inside a pyramid. Therefore, if the integral polytope*

$$C_3 = conv(w_1, w_2, ..., w_n)$$

is integrally indecomposable then P *is integrally indecomposable by [2, Theorem 4.11].*

(6) *Consider Figure 6. In order to have another example of integrally indecomposable polytope* Q, *we take an extra integral point* v_0 *and form the integral polytope* $Q = conv(C, v_0)$ *where* $C = conv(C_1, C_2, C_3)$ *is the polytope considered in case (4). Suppose that the lateral colored triangular faces of* Q *are integrally indecomposable. If the polytope*

$$C_3 = conv(w_1, w_2, ..., w_n)$$

is integrally indecomposable then so is Q. *Note that* Q *may not lie inside a pyramid.*

Note that in Example 2.5, (1), (2), (3), (4), (5) it is impossible to find a strong chain of homothetically indecomposable faces joining any two distinct vertices of the related polytopes.

Corollary 2.6 *If all 2-dimensional faces of a polytope* P *in* $\mathbb{R}^n$ *are integrally indecomposable, then so is* P.

Proof. Let $Q = conv(q_1, ..., q_m)$ be a summand of $P = conv(p_1, ..., p_m)$, where to have the same number of vertices we allow the repetition of vertices of Q. Consider any 2-dimensional face $F_P(u) = conv(p_i, p_{i+1}, ..., p_{j-1}, p_j) = P \cap H_P(u)$ of P, which is formed by the intersection of P with a supporting hyperplane $H_P(u)$ of P having normal vector $u \in \mathbb{R}^n$. Then, since $F_P(u)$ is integrally indecomposable, the face $F_Q(u) = conv(q_i, q_{i+1}, ..., q_{j-1}, q_j) = Q \cap H_P(u)$ of Q must be of the form $F_Q(u) = F_P(u) + v$ for some nonzero vector $v \in \mathbb{R}^n$. So, any edge $[q_r, q_{r+1}]$ of Q must be of the form $[q_r, q_{r+1}] = [p_r, p_{r+1}] + v$. Hence, $q_r = p_r + v$ and $q_{r+1} = p_{r+1} + v$. Since any two edges E, E' of P can be joined by a strong chain of integrally indecomposable faces $F_0, ..., F_s$, where $E \subset F_0, ..., E' \subset F_s$ and $dim\ (F_i \cap F_{i+1}) \geq 1$ we deduce that $q_i = p_i + v$ for all $i, = 1, ..., m$. So, $Q = P + v$. Consequently, P is integrally indecomposable. □

A family $\mathcal{F}$ of faces of a polytope P is called *strongly connected* if for each $F, G \in \mathcal{F}$, there exists a strong chain $F = F_1, F_2, ..., F_m = G$ with each $F_i \in \mathcal{F}$. A subset $\mathcal{F}$ of faces *touches* a face F of P if $(\bigcup_{F_i \in \mathcal{F}} F_i) \cap F \neq \emptyset$. Recall that a facet of P is a face F of dimension $dim(F) = dim(P) - 1$. See [7].

Theorem 2.7 *If P is an integral polytope having a strongly connected family of integrally indecomposable faces that touches each of its facets then it is integrally indecomposable.*

Proof. We can consider an n-dimensional polytope P in $\mathbb{R}^n$, which has a strongly connected family $\mathcal{F}$ of integrally indecomposable faces touching every facet of P. We can express P as

$$P = \{x \in \mathbb{R}^n : x \cdot u_i \leq h_P(u_i), \quad i = 1, ..., m\},$$

where $u_1, ..., u_m$ are the outer normal vectors to the facets of P having supporting functions $h_P(u_i) = sup_{x \in P}(x \cdot u_i)$ and the supporting hyperplanes $H_P(u_i) = \{x \in \mathbb{R}^n : x \cdot u_i = h_P(u_i)\}$.

Let us suppose that $P = Q + R$ for some integral polytopes Q, R in $\mathbb{R}^n$. Now, consider any strong chain $F_0, F_1, ..., F_k \in \mathcal{F}$ of integrally indecomposable faces of P. Let G_j be the face of Q corresponding to F_j, i.e. $F_j = G_j + H_j$ for some face H_j of R. Since G_j is a summand of the integrally indecomposable face F_j, there exists a vector $t_j \in \mathbb{R}^n$ such that $G_j = F_j + t_j$. Since $dim(F_{j-1} \cap F_j) \geq 1$ for each j, we see that $t_{j-1} = t_j$. Therefore, for any strongly connected family $\mathcal{F}$ of integrally indecomposable faces of P, we have a vector $t \in \mathbb{R}^n$ such that if G is the face of Q corresponding to $F \in \mathcal{F}$ then $G = F + t$.

By the hypothesis of our theorem, the family $\mathcal{F}$ touches every facet of P. If $F_i = F_P(u_i)$ is such a facet then it has a vertex a lying in some face $F_P(v) \in \mathcal{F}$. The corresponding vertex b of Q lies in $F_Q(v)$. Hence, we have $b = a + t$. By considering the support function h_Q of Q, we have $h_Q(u_i) = b \cdot u_i = (a+t) \cdot u_i = a \cdot u_i + t \cdot u_i = h_P(u_i) + t \cdot u_i = h_{P+t}(u_i)$ for $i = 1, ..., m$. As a result, $Q = P + t$, showing that P is integrally indecomposable. □

The following theorem is a consequence of Theorem 2.7.

Theorem 2.8 *Let A and B be integral polytopes such that $C = conv(A \cup B)$ with $dim(C) = dim(A) + dim(B) + 1$. Moreover, suppose also that a_i, a_{i+1}, a are vertices of A, such that a_i and a_{i+1} are adjacent, and b is a vertex of B satisfying $gcd(b - a_i, b - a_{i+1}) = 1$ or $gcd(b - a) = 1$. Then C is integrally indecomposable.*

In addition, if $C = (v_1, v_2, ..., v_n)$ then, it is integrally indecomposable if and only if $gcd(v_1 - v_2, v_1 - v_3, ..., v_1 - v_n) = 1$.

Proof. Every facet of $C = conv(A \cup B)$ contains either A or B. Therefore, if a_i, a_{i+1}, a are vertices of A, a_i and a_{i+1} are being adjacent, and b is a vertex of B with $gcd(b - a_i, b - a_{i+1}) = 1$ or $gcd(b - a) = 1$ then the face $T = conv(a_i, a_{i+1}, b)$, which is an integrally indecomposable triangle, or the face $L = conv(a, b)$, which is an integrally indecomposable line segment, meets every facet of C. Therefore, by Theorem 2.7, taking $\mathcal{F} = \{T\}$ or $\mathcal{F} = \{L\}$, C is integrally indecomposable.

Second part follows from Lemma 2.2 since C is also homothetically indecomposable by [7, Theorem 3]. □

Example 2.9 (i) Numeric example for Theorem 2.4:
Let us consider the polytopes
$C_1 = conv((0, 10, 0),\ (15, 0, 0),\ (30, 0, 0),\ (35, 10, 0),\ (35, 28, 0),$
$(32, 40, 0),\ (16, 40, 0),\ (0, 26, 0)),$

$C_2 = conv((5,12,10),\ (14,3,10),\ (27,3,10),\ (33,9,10),\ (33,24,10),$
$(28,3,10),\ (16,33,10),\ (5,26,10))$,
$C_3 = conv((12,14,20),\ (17,7,20),\ (25,7,20),\ (30,10,20),\ (30,20,20),$
$(26,23,20),\ (18,23,20),\ (12,22,20))$
lying in the planes $z = 0,\ z = 10$ and $z = 20$, respectively. Then, the polytope $C = conv(C_1, C_2, C_3, (21,15,30))$, which is of type of the polytope in Example 2.5, (6), is integrally indecomposable by Theorem 2.4 since C has a strong chain of integrally indecomposable triangular faces joining any two of its vertices and $gcd((21,15,30) - (12,14,20)) = gcd(9,1,10) = 1$.

(ii) Numeric example for item (5) of Example 2.5:
Now, take the polytopes
$P_1 = conv((0,10,0),(15,0,0),(30,12,0),(12,36,0))$,
$P_2 = conv((5,14,a),(14,6,a),(24,14,a),(11,12,a))$,
$P_3 = conv((8,14,b),(15,10,b),(20,14,b),(11,18,b))$
located in the planes $z = 0,\ z = a$ and $z = b$, respectively, with a, b positive integers such that $b \geq a$. P_3 is an integrally indecomposable quadrangle since its all edges are primitive. We see that the polytope $P = conv(P_1, P_2, P_3)$ is integrally indecomposable by Theorem 2.4 (also by Theorem 2.7) since it has a strongly connected family $\mathcal{F}$ of integrally indecomposable faces which connects any two vertices of P (which touches each facet of P.) Actually, this strongly connected family of faces is
$\mathcal{F} = \{conv((8,14,b),(15,10,b),(20,14,b),(11,18,b)),$
$conv((0,10,0),(15,0,0),(14,6,a)), conv((0,10,0),(14,6,a),(5,14,a))$
$conv((5,14,a),(14,6,a),(15,10,b), conv((5,14,a),(15,10,b),(8,14,b)),$
$conv((30,12,0),(12,36,0),(24,14,a)), conv((12,36,0),(11,22,a),(24,14,a)),$
$conv((24,14,a),(11,22,a),(20,14,b), conv((11,22,a),(20,14,b),(11,18,b))\}$.

(iii) Numeric example for item (4) of Example 2.5:
Consider the polytopes
$Q_1 = conv((5,14,c),(14,6,c),(24,14,c),(11,12,c))$,
$Q_2 = conv((0,10,d),(15,0,d),(30,12,d),(12,36,d))$,
$Q_3 = conv((8,14,e),(15,10,e),(20,14,e),(11,18,e))$
placed in the planes $z = c,\ z = d$ and $z = e$ respectively with c, d, e positive integers such that $e > d > c$. Q_3 is an integrally indecomposable quadrangle since its all edges are primitive. We see that the polytope $Q = conv(Q_1, Q_2, Q_3)$ is integrally indecomposable by Theorem 2.4 (also by Theorem 2.7) since it has a strongly connected family $\mathcal{F}$ of integrally indecomposable faces which connects any two vertices of Q (which touches each facet of Q.) The suitable strongly connected family of integrally indecomposable faces is
$\mathcal{F} = \{conv((8,14,e),(15,10,e),(20,14,e),(11,18,e)),$
$conv((0,10,d),(15,0,d),(14,6,c)), conv((0,10,d),(14,6,c),(5,14,c))$
$conv((5,14,c),(14,6,c),(15,10,e), conv((5,14,c),(15,10,e),(8,14,e)),$
$conv((30,12,d),(12,36,d),(24,14,c)), conv((12,36,d),(11,22,c),(24,14,c)),$
$conv((24,14,c),(11,22,c),(20,14,e), conv((11,22,c),(20,14,e),(11,18,e))\}$.

Example 2.10 Let $P = conv(v_1, v_2, ..., v_k)$ be an $(m-1)$-dimensional integral polytope lying in a hyperplane H in $\mathbb{R}^n$. Take any integral point $v \notin H$. Then the pyramid C=conv(P,v) is homothetically indecomposable by Theorem 2.8 since

$$m = dim(C) = dim(P) + dim(\{v\}) + 1 = (m-1) + 0 + 1.$$

In particular, our pyramid C is integrally indecomposable if and only if

$$gcd(v - v_1, v - v_2, ..., v - v_k).$$

Consequently, e.g., for two distinct integral points v_1 and v in $\mathbb{R}^n$, the line segment $\ell = [v_1, v]$ is integrally indecomposable if and only if $gcd(v - v_1) = 1$ since

$$dim(\ell) = 1 = 0 + 0 + 1 = dim(\{v_1\}) + dim(\{v\}) + 1.$$

Moreover, if v_1, v_2 and v are three distinct nonlinear integral points in $\mathbb{R}^n$ then the triangle $T = conv(v_1, v_2, v)$ is integrally indecomposable if and only if $gcd(v - v_1, v - v_2) = 1$ since

$$dim(T) = 2 = 1 + 0 + 1 = dim([v_1, v_2]) + dim(\{v\}) + 1.$$

Let $\ell_1 = [v_1, v_2]$ and $\ell_2 = [v_3, v_4]$ be the skew line segments formed by the distinct integral points v_1, v_2, v_3, v_4 in $\mathbb{R}^n$ not lying in the same plane. Then, by Theorem 2.8, the polytope $conv(v_1, v_2, v_3, v_4)$ is integrally indecomposable if and only if $gcd(v_1 - v_2, v_1 - v_3, v_1 - v_4) = 1$. Because, we have

$$dim(conv(\ell_1 \cup \ell_2)) = 3 = 1 + 1 + 1 = dim(\ell_1) + dim(\ell_2) + 1.$$

Remark 2.11 Example 2.10 provides instructive examples relevant to [4, Fact 2-page 318].

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References

[1] Ewald, G.: Combinatorial Convexity and Algebraic Geometry, GTM 168, Springer 1996.

[2] Gao, S.: Absolute irreducibility of polynomials via Newton polytopes, Journal of Algebra 237, no.2, 501-520 (2001).

[3] Gao, S., Lauder, A.G.B.: Decomposition of Polytopes and Polynomials, Discrete and Computational Geometry 26, no.1, 89-104, (2001).

[4] Grümbaum, B.: Convex Polytopes, London, New York, Sydney, Interscience Publications, 1967.

[5] Kallay, M.: Indecomposable Polytopes, Israel J. of Math. 41, no.3, 235-243 (1982).

[6] Lipkovski, A.: Newton Polyhedra and Irreducibility, Math. Z. 199, no.2, 119-127 (1998).

[7] Mcmullen, P.: Indecomposable Convex Polytopes, Israel J. of Math. 58, no.3, 321-323 (1987).

[8] Meyer, W.: Indecomposable Polytopes, Trans. A. M. S. 190, 77-86 (1974).

[9] Ostrowski, A.M.: On multiplication and factorization of polynomials I, Lexicographic orderings and extreme aggregates of terms, Aequationes Math. 13, 201-228 (1975).

[10] Shephard, G.C.: Decomposable Convex Polyhedra, Mathematika 10, 89-95 (1963).

[11] Smilansky, Z.: An Indecomposable Polytope All of Whose Facets Are Decomposable, Mathematika 33, no.2, 192-196 (1986).

[12] Smilansky, Z.: Decomposability of Polytopes and Polyhedra, Geometriae Dedicata 24, no.1, 29-49 (1987).

[13] Ziegler, G.M.: Lectures on Polytopes, GTM 152, Springer-Verlag, 1995.

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