ICAENS 2022

3rd International Conference on Applied Engineering and Natural Sciences

https://www.icaens.com/

July 20-23, 2022, Konya, Turkey

Numerical Solution of Free-Fall Equation Using Two Different Approaches of the Taylor Series Expansion

Mehmet Ekrem Çakmak

Department of Energy Systems Engineering, Yalova University, Türkiye

mehmet.cakmak@yalova.edu.tr

Abstract – The idea of using the Taylor Series Expansion to solve non-differentiable equations arose from a student challenge in the numerical methods class. In order to convince the students that the numerical methods had substantial importance in solving equations encountered in engineering design, the students were challenged to find the velocity of free-falling object at a given time using 4^{th} (or higher) order Taylor Series Expansion *assuming that* the original equation (both with linear and quadratic relation of air resistance with the velocity) could not be mathematically differentiated. Unfortunately, none of the students came up with a solution since there was no explicit solution was found over the internet. The case had led to sharing the idea and the example for the colleagues who might be willing to use them in their classes.

Keywords – TSA, numerical analysis, linear, quadratic, engineering education

I. Introduction

Maybe the most important and difficult point in engineering education is convincing the students that the engineering courses must have the practical use of mathematics as well as other natural sciences. In other words, an engineer ultimately needs the fundamental knowledge of natural sciences to develop practical use of them for humanity. Especially an engineer should use the mathematics competently. This skill, at least, provides the ability of critical thinking. But at first, the prejudices against mathematics acquired by the students before undergraduate education (i.e., college) should be broken. Then, the bloom of the discovery of the linkage between the theory and practice flourishes among the students.

To achieve this target, finding examples that could challenge and provoke the students' minds is substantial. Preferably, the answers of these in class examples should not be found over the internet, at least explicitly. Thus, the idea of challenging engineering students to find usage of the Taylor Series Expansion (TSA) for calculating value of a non-differentiable function at any given point has arisen from this perception. Furthermore, it is suggested that in order to follow the error caused by the numerical approach, the function of interest should be originally differentiable, however, the students are instructed to assume that the given function cannot be differentiated. At the end, the comparison of analytical and numerical approaches may also help the students understand the differences between these approaches.

In this study, an example of the idea mentioned above is presented for our colleagues for their appraisal.

II. Method

The equation chosen for the idea was the function which is derived to calculate the velocity of free-falling object ([1]-[3]). Along with linear relation of the air resistance with velocity (Equation 1) ([1]), nonlinear relation (Equation 2) ([2]) is considered to

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x_{i+1} - x_i)^3 + \frac{f^{(4)}(x_i)}{4!}(x_{i+1} - x_i)^4 \quad (4)$$

demonstrate the effect of nonlinearity on the error caused by the numerical approach.

$$\frac{dv}{dt} = g - \frac{c}{m}v \quad (1)$$

$$\frac{dv}{dt} = g - \frac{c}{m}v^2 \quad (2)$$

where g is the gravitational acceleration (9.81 m/s^2), c is the drag coefficient (taken as 12.5 kg/s for linear, 0.25 kg/m for quadratic), m is the mass of the free-falling object (68.1 kg).

Here it should be noted that "c" the drag coefficient in both equations is called "lumped" coefficient. It can be detailed to involve the effect of the cross-sectional area of free-falling object etc., as in [3].

In this study, along with the analytical solutions, two different numerical approaches were used, namely, Method I and Method II. The detailed analytical solutions of Equations 1 and 2 can be found in [1] and [2].

The errors found in numerical solutions are represented with the true error (Equation 3).

$$the\ true\ error = \frac{|analytical\ value - numerical\ value|}{analytical\ value}\, 100\% \quad (4)$$

A. Method I: TSA Solutions including Analytical Solutions

The equations (1 and 2) were solved using 4th order TSA (Equation 4) in which high order derivatives of the functions were obtained from the analytical derivatives of Equations 5 and 6 (Table 1 and 2).

$$v(t) = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right) \quad (5)$$

$$v(t) = \sqrt{\frac{gm}{c}}\tanh\left(\sqrt{\frac{gc}{m}}t\right) \quad (6)$$

Table 1. Analytical Derivatives of Equation 5

High Order Derivatives
$v'(t) = ge^{-\left(\frac{c}{m}\right)t}$
$v''(t) = -\frac{cg}{m}e^{-\left(\frac{c}{m}\right)t}$
$v^{(3)}(t) = \frac{c^2 g}{m^2}e^{-\left(\frac{c}{m}\right)t}$
$v^{(4)}(t) = -\frac{c^3 g}{m^3}e^{-\left(\frac{c}{m}\right)t}$

Table 2. Analytical Derivatives of Equation 6

High Order Derivatives
$v'(t) = gsech^2\left(\sqrt{\frac{gc}{m}}t\right)$
$v''(t) = -2g^{3/2}\sqrt{\frac{c}{m}}tanh\left(\sqrt{\frac{gc}{m}}t\right)sech^2\left(\sqrt{\frac{gc}{m}}t\right)$
$v^{(3)}(t) = -2\frac{c}{m}g^2 sech^2\left(\sqrt{\frac{gc}{m}}t\right)\left[sech^2\left(\sqrt{\frac{gc}{m}}t\right) - 2tanh^2\left(\sqrt{\frac{gc}{m}}t\right)\right]$
$v^{(4)}(t) = \frac{8\,g^4}{\left(\frac{gm}{c}\right)^{3/2}}tanh\left(\sqrt{\frac{gc}{m}}t\right)sech^2\left(\sqrt{\frac{gc}{m}}t\right)\left[2sech^2\left(\sqrt{\frac{gc}{m}}t\right) - tanh^2\left(\sqrt{\frac{gc}{m}}t\right)\right]$

B. Method II: TSA Solutions using the Idea

The idea of the current study may be explained as following: The equations 1 and 2 were solved by using 4th order TSA as if they could not be mathematically differentiated. Namely, high order derivatives of the velocity function with respect to time, involved in Equation 4, were derived to be function of the first derivative, which also included the initial value, and the initial value of the function itself (Tables 3 and 4).

Table 3. Derivatives obtained from Equation 1 as a function the first derivative and the initial value of the function itself

High Order Derivatives
$\frac{dv}{dt} = v' = g - \frac{c}{m} v(t)$
$v''(t) = -\frac{c}{m} v'(t)$
$v^{(3)}(t) = \frac{c^2}{m^2} v'(t)$
$v^{(4)}(t) = -\frac{c^3}{m^3} v'(t)$

Table 4. Derivatives obtained from Equation 2 as a function the first derivative and the initial value of the function itself

High Order Derivatives
$\frac{dv}{dt} = v' = g - \frac{c}{m} v^2(t)$
$v''(t) = -\frac{2c}{m} v'(t) v(t)$
$v^{(3)}(t) = -\frac{2gc}{m} v'(t) + \frac{6\,c^2}{m^2} v'(t) v^2(t)$
$v^{(4)}(t) = \frac{8\,c^2}{m^2} v'(t) v(t) \left(2 - \frac{3\,c}{m} v^2(t)\right)$

In addition, the effects of initial value "$f(x_i)$" and the step size "$(x_{i+1} - x_i)$" on the results had also been tested.

III. Results

The velocity at t=5 seconds were calculated analytically and numerically for comparison.

Analytically calculated values of velocity at t=5 s for linear and quadratic relations were found to be 32.0985 m/s and 38.2154 m/s, respectively.

A. *The Results for Linear Relation*

The values of velocity at t=5 s. calculated by Method I and II were the same, 31.8476 m/s with a true error of 0.7815%, where the initial value was v(0)=0 and step size was 5.

Keeping the same step size but changing the initial value from v(0.1)=0.9720 to v(1)=8.9623, the true error went down to 0.2191% from 0.6955% for both methods.

As the step size of 0.2 was used, the true error went down to 0.0000% with the initial value of v(0)=0.

The results obtained by both methods were the same for the solution of Equation 1.

B. *The Results for Quadratic Relation*

The values of velocity at t=5 s. calculated by Method I and II were 9.2220 m/s and 34.3296 m/s, with the true errors of 75.8683% and 10.1682%, respectively, in which the initial value and step size were taken as v(0)=0 and 5.

Using Method I the velocity values, as keeping the same step size but changing the initial value from v(0.1)=0.9809 to v(1)=9.6939, the true error went down to 50.2887% from 73.1630%.

Using Method II the velocity values, as keeping the same step size but changing the initial value from v(0.1)=0.9809 to v(1)=9.6939, the true error went down to 6.4203% from 9.9852%.

As the step size of 0.2 was used for Method I, the true error went down to 1.4784% with the initial value of v(0)=0.

As the step size of 0.2 was used for Method II, the true error went down to 0.8657% with the initial value of v(0)=0.

IV. DISCUSSION

As expected for Equation 1, the results are virtually the same calculated with Method I and II. The linear relation involved in the equation has led this observation. In addition, because of its linearity, the calculations with both numerical methods do not require smaller step sizes. In other words, the numerical solutions of Equation 1 with both methods are not sensible to the step size. Although, it is expected that the calculations with Method II would be sensitive to the initial value of v(0)=0 since it involves the function "*v(t)*" itself, it is found that the effect of the initial value is negligible for the numerical solutions of Equation 1 with both methods.

Because of its non-linear nature, it is observed that the numerical solutions of Equation 2 are sensitive both to the initial value and step size. Interestingly, more accurate results are expected in the numerical calculations of Equation 2 with Method II since it involves the analytical solution of the Equation 2, but Method II produces better results with the same initial value and step size.

V. CONCLUSIONS

Compared to Method I, Method II creates promising results even for non-linear equations.

Method I is more vulnerable to step size than Method II.

REFERENCES

[1] S. C. Chapra and R. P. Canale, *Numerical Methods for Engineers*, 7th ed., New York, NY: McGraw-Hill, 2015.
[2] S. C. Chapra, *Applied Numerical Methods with MATLAB® For Engineers and Scientists*, 3rd ed., New York, NY: McGraw-Hill, 2012.
[3] P. Timmerman and J. P. van der Weele, “On the rise and fall of a ball with linear and quadratic drag”, *Am. J. Phys.*, vol. 67 (6), pp. 538–546, June 1999.